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On supersingular rank one perturbations of the selfadjoint operators (Spectral and Scattering Theory and Related Topics)

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On supersingular rank one perturbations of the selfadjoint operators

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1 Introduction.

We shall consider the singular rank one perturbations of the positive selfadjoint operator. First we shall recall the notation of singular rank one perturbation. Let H be a Hilbert space, A a (positive) selfadjoint operator and $H_s := \{u; \|(1 + |A|)^{s/2}u\| < \infty\}$. Assume that $\varphi \in H_{-n} \setminus H_{-n+1}$. We shall put

$$A_\alpha = A + \alpha \langle \cdot, \varphi \rangle \varphi. \quad (1)$$

We call $\langle \cdot, \varphi \rangle \varphi$ “(resp. super)singular rank one perturbation” of A for $n = 1, 2$ (resp. $n \geq 3$). The main purpose is to construct a operator \mathcal{A} corresponding to A_α . We shall give a method of the construction of a Hilbert space \mathcal{H} and an operator \mathcal{A} in \mathcal{H} for $n = 3$. (section 2). For $n = 1, 2$ the operator A_α is recognized as a selfadjoint operator by using “restriction and extension theory”. (See §2).

Next we shall consider the supersingular rank one perturbation for the selfadjoint operator. There are two approaches for the problem:

1. Using Pontryagin space (Krein space):

A. Dijksma, H. Langer, Yu. Shondin and C. Zeinstra [9], J.F.van Diejen and A.Tip

2.

3. Using Hilbert space:

I. Andronov ([3]), P. Kurasov and K. Watanabe ([16], [17]), P. Kurasov ([15]).

1. In general the norm of the Pontryagin space \mathcal{P} is not positive definite, but they can construct selfadjoint operator \mathcal{A} in \mathcal{P} corresponding A_α .
2. We can consider the operator \mathcal{A} corresponding to A_α in the new Hilbert space, but \mathcal{A} is not selfadjoint except for $n = 3$.

Example 1 In [3] he considered the operator

$$A = -\Delta \text{ in } L^2(\mathbb{R} \times \mathbb{R}_+)$$

(Neumann condition) and

$$\varphi = \partial_{x_2} \delta(x_1, x_2).$$

The author can not give completely the articles related to the singular perturbation theory. Many references can be seen in [2].

2 H_{-1} - and H_{-2} -perturbation.

In this section we shall review the singular rank one perturbation. (H_{-1} , H_{-2} -perturbations). Let $\varphi \in H_{-n} \setminus H_{-n+1}$ ($n = 1, 2$) and A^0 the restriction of A to the space

$$\begin{aligned} D(A^0) &= \{u \in D(A); \langle u, \varphi \rangle = 0\}, \\ A^0 u &= Au, \quad u \in D(A^0). \end{aligned}$$

Then we shall consider the relation between the operator A_α and the (von Neumann)extension $A(\theta)$ of A_0 . Using “restriction and extension theory” we recognize A_α as a selfadjoint operator.

Two extension methods:

(I) Direct extension:

$$\begin{aligned} D(A_\alpha) \ni U &= u + u_1 \frac{A}{A^2 + 1} \varphi, \quad u \in H_2, u_1 \in \mathbb{C}, \\ \langle u, \varphi \rangle &= -\left(\frac{1}{\alpha} + \left\langle \frac{A}{A^2 + 1} \varphi, \varphi \right\rangle\right) u_1, \end{aligned} \tag{2}$$

$$A_\alpha U := Au - u_1 \frac{1}{A^2 + 1} \varphi, \tag{3}$$

where, if $\varphi \in H_{-2} \setminus H_{-1}$, then we put

$$\left\langle \frac{A}{A^2 + 1} \varphi, \varphi \right\rangle \equiv c \in \mathbb{R},$$

(II) von Neumann extension: for $\theta \in [0, \pi)$,

$$D(A(\theta)) \ni U = \tilde{u} + \tilde{u}_1 \frac{\sin \theta A - \cos \theta}{A^2 + 1} \varphi, \quad \tilde{u} \in D(A^0), \tilde{u}_1 \in \mathbb{C}, \quad (4)$$

$$A(\theta)U = A^0 \tilde{u} + \tilde{u}_1 \frac{-\cos \theta A - \sin \theta}{A^2 + 1} \varphi. \quad (5)$$

Theorem 1 Let $\varphi \in H_{-1} \setminus H$. Then there exists a bijection between A_α and $A(\theta)$, i.e., if the relation of α and θ , $\theta \in [0, \pi)$, is

$$\left\langle \frac{1}{A^2 + 1} \varphi, \varphi \right\rangle \cos \theta - \left(\frac{1}{\alpha} + \left\langle \frac{A}{A^2 + 1} \varphi, \varphi \right\rangle \right) \sin \theta = 0, \quad (6)$$

then $A_\alpha = A(\theta)$. For $n = 2$ we put $c \in \mathbb{R}$ in (6) instead of $\left\langle \frac{A}{A^2 + 1} \varphi, \varphi \right\rangle$.

Proof. (i) for $A(\theta)$. (von Neumann's method. cf. [21]) Using the deficiency elements $h_{\pm i} = \frac{1}{A \mp i} \varphi$, we put

$$U = \tilde{u} + \frac{\tilde{u}_1}{2} (h_i - e^{i2\theta} h_{-i}), \quad \tilde{u} \in D(A^0), \tilde{u}_1 \in \mathbb{C},$$

and define

$$A(\theta)U = A^0 \tilde{u} + i \frac{\tilde{u}_1}{2} (h_i + e^{i2\theta} h_{-i}). \quad (7)$$

Then $A(\theta)$ is the selfadjoint extension of A^0 . In particular, $A(0) = A$.

(ii) for A_α . (cf. [2]). The domain of A_α is $(A_\alpha U \in H)$

$$\begin{cases} U = u + u_1 \frac{A}{A^2 + 1} \varphi, & u \in D(A), u_1 \in \mathbb{C} \\ \langle u, \varphi \rangle = -\left(\frac{1}{\alpha} + \left\langle \varphi, \frac{A}{A^2 + 1} \varphi \right\rangle\right) u_1. \end{cases} \quad (8)$$

This means that there is a linear relation in $D(A) \oplus \{a \frac{A}{A^2 + 1} \varphi\}$.

(iii) Rewriting the element of $D(A(\theta))$ and substituting it to (8), we obtain Theorem.

3 H_{-3} -perturbation

We assume

$$A \geq 0.$$

We shall consider the case of

$$\varphi \in H_{-3} \setminus H_{-2}$$

and construct the operator corresponding to the operator

$$A_\alpha = A + \alpha \langle \cdot, \varphi \rangle \varphi$$

in the extended Hilbert space in

$$\mathcal{H} = H_1 \dot{+} \mathbb{C}.$$

Remark 1 *If we restrict A to*

$$D(A^0) = \{u \in H_3; \langle u, \varphi \rangle = 0\},$$

then A^0 is essentially selfadjoint in H . So any selfadjoint extension of A^0 is A .

Let a_1 be a positive constant and put

$$g_1 = \frac{1}{A + a_1} \varphi.$$

To construct the extended Hilbert space \mathcal{H} (suitable for A_α) we put

$$\mathcal{H}_{pre} = \text{Dom}(A^0) \dot{+} \mathbb{C} \ni \mathcal{U} = (u, u_1). \quad (9)$$

Note that $\mathcal{H}_{pre} \subset H_3 \dot{+} \mathbb{C}$. We define the following natural embedding ρ of the space \mathcal{H}_{pre} into the space H_{-1} :

$$\begin{aligned} \rho &: \mathcal{H}_{pre} \rightarrow H_{-1} \\ (u, u_1) &\mapsto u + u_1 g_1. \end{aligned} \quad (10)$$

Then the scalar product in the space \mathcal{H}_{pre} can be introduced using the following formal calculations where b is a certain positive constant:

$$\begin{aligned} \ll \mathcal{U}, \mathcal{V} \gg &= \langle \rho \mathcal{U}, \rho \mathcal{V} \rangle + b \langle \rho \mathcal{U}, A \rho \mathcal{V} \rangle \\ &= \langle u + u_1 g_1, v + v_1 g_1 \rangle + b \langle u + u_1 g_1, A(v + v_1 g_1) \rangle \\ &= \langle u, v \rangle + b \langle u, Av \rangle + \bar{u}_1 v_1 (\|g_1\|^2 + b \langle g_1, Ag_1 \rangle) \\ &\quad + \bar{u}_1 (\langle g_1, v \rangle + b \langle Ag_1, v \rangle) + v_1 (\langle u, g_1 \rangle + b \langle u, Ag_1 \rangle). \end{aligned}$$

The last two terms can be simplified taking into account that

$$Ag_1 = -a_1g_1 + \varphi$$

and the fact of $u, v \in H_3 \cap D(A^0)$. Then the scalar product is given by the expression

$$\begin{aligned} \ll \mathcal{U}, \mathcal{V} \gg &= \langle u, v \rangle + b\langle u, Av \rangle + \bar{u}_1v_1 (\|g_1\|^2 + b\langle g_1, Ag_1 \rangle) \\ &+ (1 - ba_1) (u_1\langle g_1, v \rangle + v_1\langle u, g_1 \rangle), \end{aligned}$$

which can be considered only formally, since the scalar product $\langle g_1, Ag_1 \rangle$ and the norm $\|g_1\|^2$ are not defined (since φ is an element from $H_{-3} \setminus H_{-2}$). To define the scalar product we extend φ as a bounded linear functional using the equalities

$$\langle g_1, g_1 \rangle = c_1, \quad \langle g_1, Ag_1 \rangle = c_2, \quad (11)$$

where c_1 and c_2 are arbitrary positive real constants. In what follows we are going to use the notation

$$d = c_1 + bc_2 \in \mathbf{R}_+. \quad (12)$$

The scalar product determined by the following expression will also be considered:

$$\ll \mathcal{U}, \mathcal{V} \gg = \langle u, v \rangle + b\langle u, Av \rangle + d\bar{u}_1v_1 + (1 - ba_1) \{ \bar{u}_1\langle g_1, v \rangle + v_1\langle u, g_1 \rangle \}. \quad (13)$$

This formula defines a sesquilinear form on the domain $\text{Dom}(A^0) + \mathbf{C}$. This form defines a scalar product only if it is positive definite.

Let us denote by $\|\mathcal{U}\|^2 = \ll \mathcal{U}, \mathcal{U} \gg$ the norm associated with the previously introduced scalar product. The space \mathcal{H} with this norm is not complete, and the following lemma describes its completion with respect to this norm.

Theorem 2 *Let $\varphi \in H_{-3} \setminus H_{-2}$, $a_1 > 0$, $g_1 := (A + a_1)^{-1}\varphi$, and*

$$\mathcal{H} = H_1 + \mathbf{C}.$$

For $\mathcal{U} = (u, u_1), \mathcal{V} = (v, v_1) \in \mathcal{H}$ we define

$$\begin{aligned} \ll \mathcal{U}, \mathcal{V} \gg &= \langle u, v \rangle + b \langle Au, v \rangle + du_1 \bar{v}_1 \\ &+ (1 - ba_1) \{u_1 \langle g_1, v \rangle + \bar{v}_1 \langle u, g_1 \rangle\} \end{aligned} \quad (14)$$

where $b > 0, d > 0$.

If we assume that

$$d > |1 - ba_1|^2 \langle (1 + bA)^{-1} g_1, g_1 \rangle, \quad (15)$$

then $\ll \cdot, \cdot \gg$ is a scalar product on \mathcal{H} and the norm induced from $\ll \cdot, \cdot \gg$ is equivalent to the standard norm of $H_1 \oplus \mathbb{C}$, i.e.,

$$\ll \mathcal{U}, \mathcal{U} \gg \cong ((1 + b'A)u, u) + d'|u_1|^2.$$

We omit the proof.

Next we define an operator \mathcal{A} in \mathcal{H} . Let $a_2 > 0$,

$$g_2 = (A + a_2)^{-1} g_1$$

and define

$$\text{Dom}(\mathcal{A}) \ni \mathcal{U} = (u_r + u_2 g_2, u_1), \quad (16)$$

$$\mathcal{A}\mathcal{U} = (Au_r - a_1 u_1 g_2, u_2 - a_1 u_1) \quad (17)$$

where $u_r \in H_3$ and $u_2 \in \mathbb{C}$.

Theorem 3 For $0 \leq \theta < \pi$ we define the linear subspace $D(\theta)$ of $\text{Dom}(\mathcal{A})$ as follows:

$$D(\theta) \ni \mathcal{U} = (u_r + u_2 g_2, u_1)$$

if and only if

$$b \sin \theta \langle \varphi, u_r \rangle - (a \sin \theta + c \cos \theta) u_1 + b \cos \theta u_2 = 0, \quad (18)$$

where a, c are constants determined by a_1, a_2, b, d, φ . Let $\mathcal{A}(\theta)$ be the restriction of \mathcal{A} to $D(\theta)$. Then $\mathcal{A}(\theta)$ is a selfadjoint operator with $\text{Dom}(\mathcal{A}(\theta)) =$

Remark 2 Using the natural embedding map ρ we have

$$\rho \mathcal{A} \mathcal{U} = Au_r - a_2 u_2 g_2 + (u_2 - a_1 u_1) g_1 \pmod{\varphi}$$

for

$$D(\mathcal{A}) \ni \mathcal{U} = (u_r + u_2 g_2, u_1).$$

Because of

$$\begin{aligned} Ag_1 &= \frac{A}{A+a_1} \varphi = \varphi - a_1 g_1, \\ Ag_2 &= g_1 - a_2 g_2. \end{aligned}$$

Proof of Theorem 3.

Step 1: \mathcal{A} is symmetric on $D(\theta)$.

We can calculate the boundary form as follows:

$$\ll \mathcal{U}, \mathcal{A} \mathcal{V} \gg - \ll \mathcal{A} \mathcal{U}, \mathcal{V} \gg = \quad (19)$$

$$\left\langle \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} \langle u_r, \varphi \rangle \\ u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} \langle v_r, \varphi \rangle \\ v_1 \\ v_2 \end{pmatrix} \right\rangle.$$

We assume that the boundary form is written as

$$\alpha \langle u_r, \varphi \rangle + \beta u_2 + \gamma u_1 = 0. \quad (20)$$

Combining above two conditions, we have

$$\begin{aligned} (19) = 0 \text{ (}\mathcal{A} \text{ symmetric)} &\iff \alpha, \beta, \gamma \in \mathbb{R}, \alpha a + \beta b + \gamma c = 0 \\ &\iff \alpha, \beta, \gamma \in \mathbb{R}, (a, b, c) \perp (\alpha, \beta, \gamma). \end{aligned} \quad (21)$$

Hence we can represent (α, β, γ) by one parameter θ : i.e.,

$$(\alpha, \beta, \gamma) = (b \sin \theta, -a \sin \theta - c \cos \theta, b \cos \theta). \quad (22)$$

Therefore $\mathcal{A}(\theta)$ is symmetric on $D(\theta)$.

Step 2: $\mathcal{A}(\theta)$ is self-adjoint on $D(\theta)$.

For $\lambda \ll 0$ we prove

$$\mathcal{R}(\mathcal{A}(\theta) - \lambda) = \mathcal{H},$$

i.e. that for any $\mathcal{V} = (v, v_1) \in \mathcal{H}$ there exists an element $\mathcal{U} = (u_r + u_2 g_2, u_1) \in \text{Dom}(\mathcal{A}(\theta))$ such that

$$(\mathcal{A}(\theta) - \lambda)\mathcal{U} = \mathcal{V}.$$

The last equation can be written as

$$\begin{cases} (A - \lambda)u_r - (a_2 + \lambda)u_2 g_2 = v; \\ u_2 - (a_1 + \lambda)u_1 = v_1. \end{cases}$$

The first of these equations can be rewritten as

$$u_r - (a_2 + \lambda)u_2 \frac{1}{A - \lambda} g_2 = \frac{1}{A - \lambda} v,$$

which implies

$$\langle u_r, \varphi \rangle - (a_2 + \lambda) \langle \frac{1}{A - \lambda} g_2, \varphi \rangle u_2 = \langle \frac{1}{A - \lambda} v, \varphi \rangle.$$

$\mathcal{U} = (u_r + u_2 g_2, u_1)$ should satisfy the boundary condition (18). Hence $(\langle u_r, \varphi \rangle, u_1, u_2) \in \mathbb{C}^3$ solves the system of linear equations

$$\begin{pmatrix} 1 & 0 & -(a_2 + \lambda)\Phi_\lambda \\ 0 & -(a_1 + \lambda) & 1 \\ b \sin \theta & -a \sin \theta - c \cos \theta & b \cos \theta \end{pmatrix} \begin{pmatrix} \langle u_r, \varphi \rangle \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \langle \frac{1}{A - \lambda} v, \varphi \rangle \\ v_1 \\ 0 \end{pmatrix}. \quad (23)$$

where we put $\Phi_\lambda = \langle \frac{1}{A - \lambda} g_2, \varphi \rangle$. The determinant of this system is:

$$-(a_1 + \lambda)b \cos \theta - b \sin \theta (a_1 + \lambda)(a_2 + \lambda)\Phi_\lambda + a \sin \theta + c \cos \theta. \quad (24)$$

(i) $\theta = 0$: Since $b \neq 0$, $\exists \lambda$ such that $-a_1 b + c - b\lambda \neq 0$.

(ii) $\theta \neq 0$: We can prove

$$\lim_{\lambda \rightarrow -\infty} |\lambda \Phi_\lambda| = \infty$$

because $\varphi \in H_{-3}$. Hence the 2-nd term of (24) is dominant of the determinant. Therefore Theorem has been proved.

Remark 3 From the proof of Theorem we know:

- (i) $\mathcal{A}(\theta)$ is semibounded from below.
- (ii) For $\theta = 0$: The solution of the linear system (23) is given by

$$\begin{aligned}\langle u_r, \varphi \rangle &= \frac{(c - b(a_1 + \lambda)) \langle \frac{1}{A-\lambda} v, \varphi \rangle + (a_2 + \lambda) \langle \frac{1}{A-\lambda} g_2, \varphi \rangle c v_1}{(c - b(a_1 + \lambda)) + (a - b(a_1 + \lambda)(a_2 + \lambda) \langle \frac{1}{A-\lambda} g_2, \varphi \rangle)}; \\ u_1 &= \frac{b v_1}{c - b(a_1 + \lambda)}; \\ u_2 &= \frac{c v_1}{c - b(a_1 + \lambda)}.\end{aligned}$$

Then the resolvent can be calculated as

$$\frac{1}{\mathcal{A}(\theta) - \lambda} (v, v_1) = \left(\frac{1}{A - \lambda} v + \left(\frac{1}{A - \lambda} g_1 \right) u_2, u_1 \right). \quad (25)$$

- (iii) $\mathcal{V} = (v, 0) \in H_1 \dagger \mathbb{C}$:

$$\begin{aligned}\rho_{\frac{1}{\mathcal{A}(\theta) - \lambda}}|_{H_1} v &= \frac{1}{A - \lambda} v + \\ &+ \frac{b \sin \theta \times}{\cos \theta (c - b(a_1 + \lambda)) + \sin \theta (a - b(a_1 + \lambda)(a_2 + \lambda) \langle \frac{1}{A-\lambda} g_2, \varphi \rangle)} \times \\ &\times \left\langle \frac{1}{A - \lambda} v, \varphi \right\rangle \left(\frac{1}{A - \lambda} \varphi \right).\end{aligned} \quad (26)$$

- (iv) $\theta = 0, v_1 = 0$.

$$\frac{1}{A - \lambda} \rho|_{H_1} = \rho_{\frac{1}{\mathcal{A}(0) - \lambda}}|_{H_1}. \quad (27)$$

Hence

$$\text{Dom}(A) \subset \text{Dom}(\mathcal{A}(0))$$

and the action coincides

$$\mathcal{A}(0)|_{\text{Dom}(A)} = A.$$

Therefore the operator $\mathcal{A}(0)$ should be considered as an **unperturbed operator**, since this is the unique operator possessing the properties described above. All of the other operators $\mathcal{A}(\theta)$ corresponding to $\theta \neq 0$ are perturbations of $\mathcal{A}(0)$.

4 H_{-n} -perturbation ($n \geq 4$).

$\varphi \in H_{-n} \setminus H_{-n+1}$ ($n \geq 4$). For simplicity we confine $n = 4$. For general n see [15]. We put

$$\begin{cases} g_1 = (A+1)^{-1}\varphi, & g_2 = (A+1)^{-1}g_1, \\ g_3 = (A+1)^{-1}g_2 \end{cases}$$

and Hilbert space and the scalar product

$$\begin{aligned} \mathcal{H} &= H_2 \oplus \mathbb{C}^2 \ni \mathcal{U} = (u, u_2, u_1), \mathcal{V} = (v, v_2, v_1), \\ \ll \mathcal{U}, \mathcal{V} \gg &= \langle (A+1)^2 u, u \rangle + u_2 \bar{v}_2 + u_1 \bar{v}_1 \end{aligned}$$

We define the maximal operator \mathcal{A} in \mathcal{H} corresponding to A_α as follows:

$$\begin{aligned} \text{Dom}(\mathcal{A}) &= \{ \mathcal{U} = (U_r + u_3 g_3, u_2, u_1) ; \\ &U_r \in H_4, u_3, u_2, u_1 \in \mathbb{C} \} \end{aligned}$$

and

$$\mathcal{A} \begin{pmatrix} U_r + u_3 g_3 \\ u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} AU_r - u_3 g_3 \\ u_3 - u_2 \\ u_2 - u_1 \end{pmatrix}. \quad (28)$$

Definition 1 Let T be a densely defined closed operator in a Hilbert space.

$$T \text{ is regular} \iff D(T) = D(T^*) \quad (29)$$

Theorem 4 For $\theta \in [0, \pi)$ let $\mathcal{A}(\theta)$ be a restriction of \mathcal{A} to

$$\sin \theta \langle U_r, \varphi \rangle + \cos \theta u_3 - \sin \theta u_2 = 0. \quad (30)$$

Then $\mathcal{A}(\theta)$ is regular. Conversely any regular restriction of \mathcal{A} is given by (30) for some $\theta \in [0, \pi)$.

Remark 4 (i) *The action of the operator $\mathcal{A}(\theta)^*$ is given by*

$$\mathcal{A}(\theta)^* \begin{pmatrix} V_r + v_3 g_3 \\ v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} AV_r - v_3 g_3 \\ v_3 + v_1 - v_2 \\ -v_1 \end{pmatrix}. \quad (31)$$

The real and imaginary parts of the operator $\mathcal{A}(\theta)$ are given by

$$\mathcal{A}(\theta) = \Re \mathcal{A}(\theta) + i \Im \mathcal{A}(\theta); \quad (32)$$

$$(\Re \mathcal{A}(\theta)) \begin{pmatrix} U_r + u_3 g_3 \\ u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} AU_r - u_3 g_3 \\ u_3 - u_2 + \frac{1}{2}u_1 \\ \frac{1}{2}u_2 - u_1 \end{pmatrix};$$

$$\Im \mathcal{A}(\theta) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}.$$

The imaginary part of $\mathcal{A}(\theta)$ is a bounded operator.

(ii) *We can prove that the spectrum of the regular operator $\mathcal{A}(\theta)$ is pure real even if the operator is not self-adjoint.*

5 Further Results and Problems.

In section 3 we confine the case $a_1, a_2, b, d = 1$. Then the Hilbert space \mathcal{H} and the scalar product are $\mathcal{H} = H_1 \oplus \mathbb{C}$ and $\ll \mathcal{U}, \mathcal{V} \gg = \langle (1+A)u, v \rangle + u_1 \bar{v}_1$, respectively. And the condition of the element of $\mathcal{A}(\theta)$ is given by, for $\mathcal{U} = (u_r + u_2 g_2, u_1) \in D(\mathcal{A}(\theta))$

$$\sin \theta \langle u_r, \varphi \rangle - \sin \theta u_1 + \cos \theta u_2 = 0, \quad (33)$$

and the operator acts as

$$\mathcal{A}(\theta)\mathcal{U} = (Au_r - u_2 g_2, u_2 - u_1).$$

We consider the following selfadjoint operator in $\mathbf{H}(=\mathcal{H})$:

$$\mathbf{A}\mathbf{U} = (\mathbf{A}u, -u_1), \quad u \in H_3, \quad u_1 \in \mathbb{C}.$$

Then \mathbf{A} is selfadjoint and $\mathbf{A} \geq -1$. The space $D(\mathbf{A})^*$ of the dual space $D(\mathbf{A})$ with respect to $\ll \cdot, \cdot \gg$ is $D(\mathbf{A})^* = H_{-1} \oplus \mathbb{C}$. We consider the rank one perturbation for \mathbf{A} .

$$\mathbf{A}_\alpha = \mathbf{A} + \alpha \ll \cdot, \mathbf{G}_1 \gg \mathbf{G}_1, \quad (34)$$

where $\mathbf{G}_1 = (g_1, -1) \in H_{-1} \oplus \mathbb{C}$.

We can see that \mathbf{A}_α is selfadjoint if and only if $c \in \mathbb{R}$ is real parameter and the following relation is satisfied

$$\mathbf{U} \in D(\mathbf{A}_\alpha) \iff \begin{cases} \mathbf{U} = \tilde{\mathbf{U}} + a \frac{\mathbf{A}}{\mathbf{A}^2+1} \mathbf{G}_1, \tilde{\mathbf{U}} \in D(\mathbf{A}), \\ \ll \tilde{\mathbf{U}}, \mathbf{G}_1 \gg = -(\frac{1}{\alpha} + c)a, \end{cases} \quad (35)$$

where

$$\frac{\mathbf{A}}{\mathbf{A}^2+1} \mathbf{G}_1 = (\frac{A}{A^2+1} g_1, \frac{1}{2})$$

We can rewrite the above relation as follows:

$$\langle u, \varphi \rangle - u_1 = -(\frac{1}{\alpha} + c)a. \quad (36)$$

Theorem 5 *There exists one to one correspondance between \mathbf{A}_α and $\mathcal{A}(\theta)$.*

$$\{\mathbf{A}_\alpha\}_{\alpha \in \mathbb{R}} = \{\mathcal{A}(\theta)\}_{0 \leq \theta < \pi}.$$

Proof. Since

$$\begin{aligned} \frac{A}{A^2+1} g_1 &= (\frac{A}{A^2+1} - \frac{1}{A+1}) g_1 + g_2 \\ &= \frac{A-1}{(A^2+1)(A+1)} g_1 + g_2, \end{aligned}$$

the elemnet of the domain of $\mathbf{A}(\theta)$ can be written as

$$\begin{aligned} \mathbf{U} &= ((u + a \frac{A-1}{(A^2+1)(A+1)} g_1) + a g_2, u_1 + \frac{1}{2}a) \\ &= (u_\tau + a g_2, u_1 + a/2). \end{aligned}$$

Substituting this to (33) we have

$$\sin \theta \langle u + a \frac{A-1}{A^2+1} g_2, \varphi \rangle - \sin \theta (u_1 + a/2) + \cos \theta a = 0. \quad (37)$$

By (36) we obtain

$$\sin \theta a \langle \frac{A-1}{A^2+1} g_2, \varphi \rangle - \sin \theta (1/\alpha + c)a - \sin \theta a/2 + \cos \theta a = 0. \quad (38)$$

Hence

$$\{\mathbf{A}_\alpha\}_{\alpha \in \mathbb{R}} = \{\mathcal{A}(\theta)\}_{0 \leq \theta < \pi}$$

Problem. (1) In this section we began with

$$D(A^0) = \{u \in H_3; \langle u, \varphi \rangle = 0\},$$

and

$$D(\mathbf{A}^0) = \{\mathbf{U} = (u, u_1) \in D(\mathbf{A}); \ll \mathbf{U}, \mathbf{G}_1 \gg = \langle u, \varphi \rangle - u_1 = 0\}. \quad (39)$$

We would like to consider the relation of A^0 and \mathbf{A}^0 . A is considered as $A : H_s \rightarrow H_{s-2}$. Hence by

$$\begin{aligned} D(A^0) &= H_3(+\text{condition}) \oplus \{0\} \\ &\subset H_3 \oplus \mathbb{C}(+\text{condition}) = D(\mathbf{A}^0), \end{aligned}$$

we can consider that $\langle u, \varphi \rangle = 0$ and $\langle u, \varphi \rangle - cu_1 = 0$ have some relation. (Because in the case $u_1 = 0$ we can identify.) But $c = 1?$ or not?

(2) Are $\mathcal{A}(\theta)$ in \mathcal{H} the operators corresponding to the operators constructed by using the Pontryagin space?

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